OCR
RECOGNISING ACHIEVEMENT

Thursday 13 June 2013 – Morning

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 By using the substitution $t = \tan\frac{1}{2}\theta$, find $\int_0^{\frac{1}{2}\pi} \frac{1}{1+\cos\theta}\,d\theta$. **[5]**

2 **(i)** Using the definitions for $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that $\cosh^2 x - \sinh^2 x \equiv 1$. **[3]**

(ii) Hence solve the equation $\sinh^2 x = 5\cosh x - 7$, giving your answers in logarithmic form. **[5]**

3 It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{3+x}\right)$ for $x > -1$.

(i) Show that $f''(x) = \frac{1}{2(x+1)^2}$. **[6]**

(ii) Hence find the Maclaurin series for $f(x)$ up to and including the term in x^2. **[4]**

4 It is given that $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x\,dx$ for $n \geqslant 0$.

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. **[5]**

(ii) Hence find I_{11} as an exact fraction. **[3]**

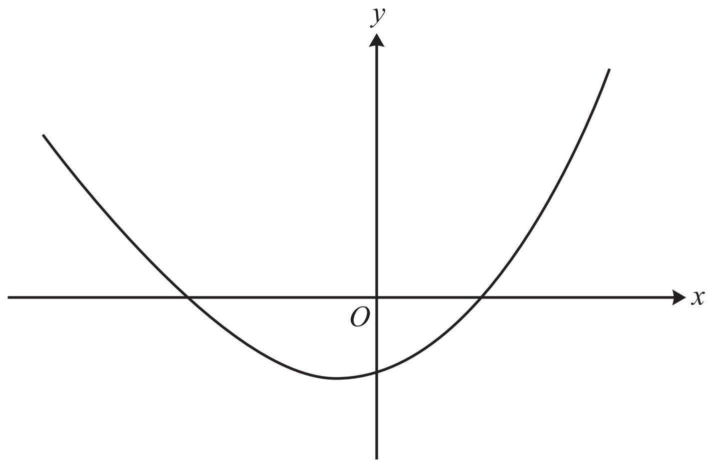
5 You are given that the equation $x^3 + 4x^2 + x - 1 = 0$ has a root, α, where $-1 < \alpha < 0$.

(i) Show that the Newton-Raphson iterative formula for this equation can be written in the form

$$x_{n+1} = \frac{2x_n^{\ 3} + 4x_n^{\ 2} + 1}{3x_n^{\ 2} + 8x_n + 1}.$$ **[3]**

(ii) Using the initial value $x_1 = -0.7$, find x_2 and x_3 and find α correct to 5 decimal places. **[3]**

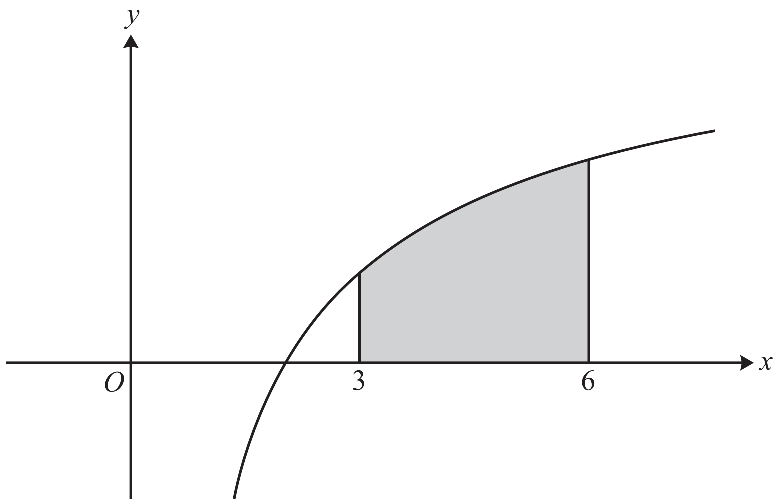
(iii) The diagram shows a sketch of the curve $y = x^3 + 4x^2 + x - 1$ for $-1.5 \leqslant x \leqslant 1$.



Using the copy of the diagram in your answer book, explain why the initial value $x_1 = 0$ will fail to find α. **[2]**

[Questions 6, 7 and 8 are printed overleaf.]

6



The diagram shows part of the curve $y = \ln(\ln(x))$. The region between the curve and the x-axis for $3 \leqslant x \leqslant 6$ is shaded.

(i) By considering n rectangles of equal width, show that a lower bound, L, for the area of the shaded region is $\frac{3}{n}\sum_{r=0}^{n-1}\ln\left(\ln\left(3+\frac{3r}{n}\right)\right)$. **[3]**

(ii) By considering another set of n rectangles of equal width, find a similar expression for an upper bound, U, for the area of the shaded region. **[1]**

(iii) Find the least value of n for which $U - L < 0.001$. **[4]**

7 The equation of a curve is $y = \frac{x^2+1}{(x+1)(x-7)}$.

(i) Write down the equations of the asymptotes. **[3]**

(ii) Find the coordinates of the stationary points on the curve. **[5]**

(iii) Find the coordinates of the point where the curve meets one of its asymptotes. **[3]**

(iv) Sketch the curve. **[3]**

8 The equation of a curve is $x^2 + y^2 - x = \sqrt{x^2+y^2}$.

(i) Find the polar equation of this curve in the form $r = \mathrm{f}(\theta)$. **[3]**

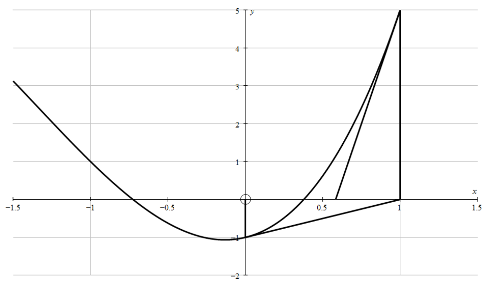
(ii) Sketch the curve. **[2]**

(iii) The line $x + 2y = 2$ divides the region enclosed by the curve into two parts. Find the ratio of the two areas. **[6]**

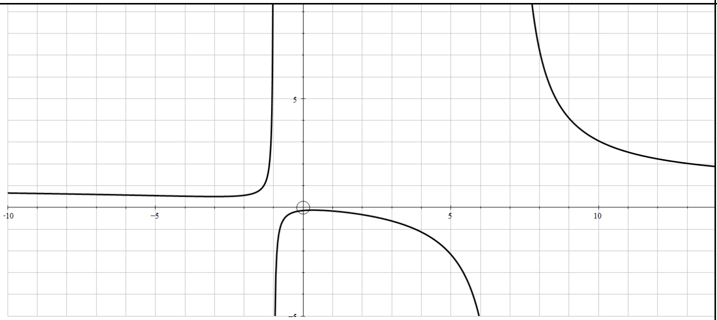
Question			Answer	Marks	Guidance	
1			$\cos\theta = \dfrac{1-t^2}{1+t^2}$	M1	Using t substitution for **both** $\cos\theta$ and $\mathrm{d}\theta$	
			$\dfrac{\mathrm{d}t}{\mathrm{d}\theta} = \dfrac{1}{2}\sec^2\dfrac{1}{2}\theta = \dfrac{1}{2}\left(1+\tan^2\dfrac{1}{2}\theta\right)$	A1	Subs correct	
			$\Rightarrow \quad \mathrm{d}t = \dfrac{1+t^2}{2}.\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = \dfrac{2\mathrm{d}t}{1+t^2}$	M1	Dealing with limits and attempting integration.	
			$\Rightarrow I = \displaystyle\int_0^1 \dfrac{1}{1+\dfrac{1-t^2}{1+t^2}}\,\dfrac{2\mathrm{d}t}{1+t^2} = \int_0^1 \dfrac{1+t^2}{1+t^2+1-t^2}\,\dfrac{2\mathrm{d}t}{1+t^2}$	A1	Correct integral	
			$\displaystyle\int_0^1 \dfrac{2\mathrm{d}t}{2} = [t]_0^1 = 1$	A1 **[5]**	Answer	
			Alternative $1+\cos\theta = 2\cos^2\frac{1}{2}\theta$ $\Rightarrow \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{1}{1+\cos\theta}\mathrm{d}\theta = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} \dfrac{1}{\cos^2\frac{1}{2}\theta}\mathrm{d}\theta = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} \sec^2\tfrac{1}{2}\theta\,\mathrm{d}\theta$ $= \dfrac{1}{2}\left[2\tan\dfrac{1}{2}\theta\right]_0^{\frac{\pi}{2}} = \tan\dfrac{\pi}{2} - \tan 0 = 1$	**SC3**		
2	(i)		$\cosh x = \dfrac{e^x + e^{-x}}{2},\ \sinh x = \dfrac{e^x - e^{-x}}{2}$	B1	Correct formulae	
			$\Rightarrow \cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2$	M1	Dealing with squaring correctly	Difference of squares can be used
			$= \dfrac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) = \dfrac{1}{4}.4 = 1$	A1 **[3]**	**www** All steps seen	

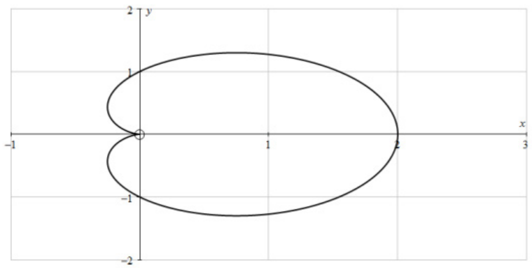
Question			Answer	Marks	Guidance	
2	(ii)		$\Rightarrow \cosh^2 x - 1 = 5\cosh x - 7$			
			$\Rightarrow \cosh^2 x - 5\cosh x + 6 = 0$	M1	Use **(i)**	
			$\Rightarrow (\cosh x - 2)(\cosh x - 3) = 0$	M1	Attempt to solve quadratic	E.g. correct formula or expansion of their brackets gives 2 out of 3 terms correct
			$\Rightarrow \cosh x = 2, 3$	A1		
			$\Rightarrow x = \cosh^{-1} 2 = \pm\ln(2 \pm \sqrt{3})$	A1	Use correct ln formula	Condone lack of ±
			and $x = \cosh^{-1} 3 = \pm\ln(3 \pm \sqrt{8})$	A1	Use correct ln formula	Condone lack of ±
				[5]		
3	(i)		$\frac{dy}{dx} = \frac{1}{1-\left(\frac{1-x}{3+x}\right)^2} \times \frac{-(3+x)-(1-x)}{(3+x)^2}$	B1	Sight of standard diffn for $\tanh^{-1} x$	
				M1	Fn of fn **and** diffn of quotient	
				A1	Soi correct quotient (i.e. correct expression for 2nd part)	
			$\Rightarrow \frac{dy}{dx} = \left(\frac{-4}{(3+x)^2 - (1-x)^2}\right) = \frac{k}{1+x}$	A1		
			$\Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)}$	A1	Correct for y'	
			$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+x)^2}$	A1	2nd diffn (NB AG)	
				[6]		

Question			Answer	Marks	Guidance	
3	(ii)		When $x=0, y=\tanh^{-1}\frac{1}{3}$ or $\frac{1}{2}\ln 2$ or $\ln\sqrt{2}$	B1	For 1st value (needs to be exact)	
			$\frac{dy}{dx}=-\frac{1}{2}$ $\frac{d^2y}{dx^2}=\frac{1}{2}$	B1	For both	
			$\Rightarrow y=\tanh^{-1}\frac{1}{3}+\left(-\frac{1}{2}\right)x+\left(\frac{1}{2}\right)\frac{x^2}{2}$	M1	Use of correct Maclaurin's series	
			$=\tanh^{-1}\frac{1}{3}-\frac{1}{2}x+\frac{x^2}{4}$	A1	Accept 0.347	
				[4]		
4	(i)		$u=\cos^{n-1}x,\ dv=\cos x\,dx$	M1*	By parts the right way round	
			$du=-(n-1)\cos^{n-2}x\sin x,\quad v=\sin x$	A1		
			$\Rightarrow I_n=\left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^2 x\ dx$	A1	Integral so far	
			$=0+(n-1)\left(I_{n-2}-I_n\right)$	*M1	Correct use of $\sin^2 x=1-\cos^2 x$ Dependent on 1st M	
			$\Rightarrow nI_n=(n-1)I_{n-2}\Rightarrow I_n=\frac{n-1}{n}I_{n-2}$	A1	**www** AG	
				[5]		
4	(ii)		$I_1=1$	B1	For I_1 soi	
			$I_{11}=\frac{10}{11}I_9=....=\frac{10}{11}.\frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}I_1$	M1	Use of **(i)** to give product of 5 fractions	
			$\Rightarrow I_{11}=\frac{3840}{10395}=\frac{256}{693}$ oe	A1	Correct fraction	
				[3]		

Question			Answer	Marks	Guidance	
5	(i)		$f(x) = x^3 + 4x^2 + x - 1$ $f'(x) = 3x^2 + 8x + 1$ $\Rightarrow x_{n+1} = x_n - \dfrac{x_n^3 + 4x_n^2 + x_n - 1}{3x_n^2 + 8x_n + 1}$ $= \dfrac{x_n\left(3x_n^2 + 8x_n + 1\right) - \left(x_n^3 + 4x_n^2 + x_n - 1\right)}{3x_n^2 + 8x_n + 1}$ $= \dfrac{2x_n^3 + 4x_n^2 + 1}{3x_n^2 + 8x_n + 1}$	B1 M1 A1 **[3]**	Diffn Correct application of N-R formula And completed with suffices on last line NB AG	
5	(ii)		$x_2 = -0.72652$, $x_3 = -0.72611$ $\Rightarrow \alpha = -0.72611$	B1 B1 B1 **[3]**		NB $x_4 = -0.726109$
5	(iii)		Sketch plus at least one tangent  Converges to another root.	B1 B1 **[2]**	At least the first tangent and vertical line to curve Or positive root or, for e.g. "$x = 0$ is the wrong side of a turning point" **www**	Use of formula to find this root numerically is not acceptable

Question			Answer	Marks	Guidance	
6	(i)		Width of rectangles is $\frac{3}{n}$	B1	Statement about width	
			$\Rightarrow$ Sum of areas of rectangles	M1	Height or area of at least one rectangle	
			$= \frac{3}{n} \times \left(\ln(\ln 3) + \ln\left(\ln\left(3 + \frac{3}{n} \right) \right) + ... \right)$	A1	Correct conclusion **www**	
			$= \frac{3}{n} \times \sum_{r=0}^{n-1} \ln\left(\ln\left(3 + \frac{3r}{n} \right) \right)$	**[3]**	1468 or	
6	(ii)		$= \frac{3}{n} \times \sum_{r=1}^{n} \ln\left(\ln\left(3 + \frac{3r}{n} \right) \right)$	B1 **[1]**		
6	(iii)		$U - L = \frac{3}{n} \times \ln(\ln 6) - \frac{3}{n} \times \ln(\ln 3)$	M1*	Subtraction to obtain the difference of two terms	
			$= \frac{3}{n}\left(\ln(\ln 6) - \ln(\ln 3) \right) = \frac{3}{n} \ln\left(\frac{\ln 6}{\ln 3} \right)$	A1		
			$\Rightarrow n > \frac{3}{0.001} \ln\left(\frac{\ln 6}{\ln 3} \right) \Rightarrow n > \frac{3}{0.001} \times \ln(1.6309)$	*M1	Dealing with inequality to obtain n dep on first M	
			$\Rightarrow$ least $n = 1468$	A1 **[4]**	Accept $n \geq 1468$ or $n > 1467$	
7	(i)		$x = -1$	B1	B1 for each	
			$x = 7$	B1		
			$y = 1$	B1 **[3]**	−1 for any extras	

Question			Answer	Marks	Guidance	
7	(ii)		$\frac{dy}{dx}=\frac{(x^2-6x-7)2x-(x^2+1)(2x-6)}{(x+1)^2(x-7)^2}$	M1 A1	Diffn using quotient rule	Or expand as partial fractions and use fn of fn rule
			$=0$ when $(x^2-6x-7)2x-(x^2+1)(2x-6)=0$			
			$3x^2+8x-3=0$	A1	Quadratic	
			$\Rightarrow x=-3,\frac{1}{3};\quad y=\frac{1}{2},-\frac{1}{8}$	A1	Both x values	Or: A1 one pair
			i.e. $\left(-3,\frac{1}{2}\right),\left(\frac{1}{3},-\frac{1}{8}\right)$	A1	Both y values	A1 other pair
				[5]		
7	(iii)		When $y=1,\ x^2-6x-7=x^2+1$	M1 A1		
			$\Rightarrow 6x=-8\Rightarrow x=-\frac{4}{3}\Rightarrow\left(-\frac{4}{3},1\right)$	A1	Coordinate pair needs to be seen.	
				[3]		
7	(iv)			B1	Left section, cutting asymptote and approaching $y=1$ from below	
				B1	Right hand section	
				B1	Middle section all below x-axis labelling intercept on graph or by a statement	
				[3]		

Question			Answer	Marks	Guidance	
8	(i)		Substitute $r^2 = x^2 + y^2$, $x = r\cos\theta$ $\Rightarrow r^2 - r\cos\theta = r \Rightarrow r = 1 + \cos\theta$	M1 A1 A1 **[3]**	 **cao**	
8	(ii)			B1 B1 **[2]**	Cardioid (General shape) Correct shape at pole, $r = 2$ and symmetric	 e.g. cusp clearly at pole, vertical tangent at $r = 2$
8	(iii)		Line cuts curve at (0, 1) and (2, 0) Total area = $2 \times \frac{1}{2} \times \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$ $= \int_0^{\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta = \int_0^{\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$ $= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi} = \frac{3}{2}\pi$ area in 1st quadrant = $\frac{1}{2} \times \int_0^{\frac{1}{2}\pi} (1+\cos\theta)^2 \, d\theta$ $= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{1}{2}\pi} = \frac{3}{8}\pi + 1$ Area under line in 1st quadrant = 1 $\Rightarrow$ Area enclosed by line and curve = $\frac{3}{8}\pi + 1 - 1 = \frac{3}{8}\pi$ $\Rightarrow$ ratio = $\left(\frac{3}{2}\pi - \frac{3}{8}\pi\right) : \frac{3}{8}\pi = 3:1$	B1 M1 A1 A1 M1 A1 **[6]**	 Formula for area used Or ratio 1 : 3	 Sight of expansion and attempt to integrate